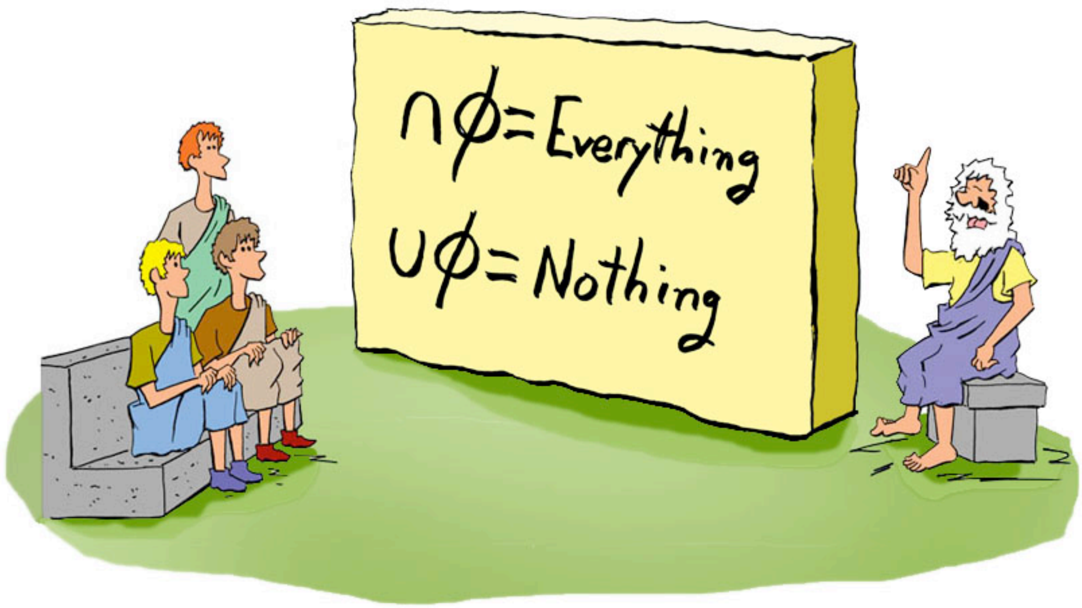
The Great Mystery of the Intersection of the Empty Set

(Explained To Leave No Doubt That Small Bits of A Bunch of Nothings Add Up To Everything)



Leaving No Doubt

Adapted from
http://www.coopertoons.com/education/emptyclass_intersection/emptyclass_union_intersection.html

Every branch of study has something that drives students nuts. In physics there's the **Clock Paradox** of the **Special Theory of Relativity**. In chemistry, it's **Pyrene** and why this compound with 4n π-electrons (that is, 16 electrons) is aromatic. Of course, both of these strange but true phenomenon have been explained in their own CooperToons **Educational Essays** at

http://www.coopertoons.com/education/specialrelativity/relativity_twinparadox.html

... and ...

http://www.coopertoons.com/education/pyreneandhuckelsrule/pyreneandhuckelsrule.html

... respectively.

But in mathematics and logic, there's one question that transcends these conundrums in sending students into spittle flinging diatribes as to what kind of dunderheads are writing their lousy textbooks.

And that's when they learn that:

The ***INTERSECTION*** of the EMPTY SET is
EVERYTHING IN THE UNIVERSE!

... or as mathematicians write:

$$\bigcap \varnothing = U$$

Ha? (to quote Shakespeare). How can you take nothing and get everything? Not only nothing, but little slices of nothing and get everything!

And then we also learn learn:

The ***UNION*** of the EMPTY SET is
ABSOLUTELY NOTHING

... which we write:

$$\cup\varnothing = \varnothing$$

That is, the union of the empty set is the ***empty set***.

True, this last statement doesn't bother people much. After all, putting together a nothing with itself should give us nothing.

But how can taking everything - the ***union*** - be ***less*** than taking bits - the ***intersection*** - of the same thing?

And to make things more confusing, there are some books that say the union and intersection of the empty set are ***both the empty set!***

How do we make sense of ***this***?

Well, you ***could*** turn to advanced textbooks for succor. But this tack is stymied by 1) the astronomically absurd prices that publishers put on their textbooks, and 2) you won't find a proof even if you ***do*** pay the ridiculous prices.

Instead you're likely to find statements like "It is up to the readers to convince themselves this is true." Or maybe the books say something like "It is often convenient to define the intersection of the empty set as the universal set."

Of course, you can always turn to the Fount of All Knowledge. There with a mouse click or two, you'll find a chat room or discussion board about the topic. Now, it's not that the answers given are necessarily wrong. But it is true they tend to leave the readers more baffled than before.

So why do college professors believe this is something the readers can easily convince themselves of? If that's true, then there should be a simple explanation. Certainly you have ***always*** wondered if there is a clear ***proof*** about the intersection and union of the empty set.

No doubt you have, as Captain Mephisto said to Sidney Brand. And if you read on you'll be able to say ...

"I Understand"
- *Manhunt of Mystery Island*, (1945)

There's one thing teachers should avoid when trying to explain the conundrums of the empty set. Don't try to make the ***math*** understandable. The math's not the problem.

Instead, the problem is one of ***logic***. More exactly, the proof will require us to transform what are nearly identical sentences in ordinary language into quite distinct ***formulas of symbolic logic***.

But we do have to start off talking about sets.

A ***set***, as everyone knows, is a collection of objects. Now for simple sets - as we will call them - the objects themselves are not sets but objects. That is, the ***elements*** of a simple set are not sets.

But there are certainly sets whose elements are sets. For instance, the set of all cats can be grouped by classes of cat.

Classes of Cat = { {Tabbies} , {Calico} , {Siamese} , {Alley} }

Clearly there's nothing wrong with grouping cats by the ***classes of cats***. And by doing so we have created a set whose elements are themselves sets.

We will see, then, that we will be talking about the ***intersection and union*** of a ***class*** of sets. That is, we're talking about a set of sets.

But get this. We'll be talking about a set of sets - but one ***that contains no sets***.

Now before you say "And from ***this***, mathematicians make a ***living***", bear with us. Things will get clearer as we go along.

But now we'll take a brief review about something you already know.

"It is common knowledge to every schoolboy ... "

- Ogden Nash
"Portrait Of The Artist As A Prematurely Old Man"

Starting with grade school, you learned about sets. That's always the first chapter of math textbooks. After that chapter you never talk about sets again for the rest of the year.

But you did learn about the ***intersection*** and ***union*** of sets. For instance, if we define the sets, A_1, A_2, and A_3 as:

$$\mathbf{A_1 = \{1, 2, 3, 4\}}$$

$$\mathbf{A_2 = \{2, 5, 6, 8\}}$$

$$\mathbf{A_3 = \{2, 8, 14, 20\}}$$

Then the *union* of A_1 , A_2 , and A_3 is:

$$\mathbf{A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 8, 14, 20\}}$$

That is we pick out the elements that you find in *at least one of the sets*.

And the intersection ....?

The *intersection* of A_1 , A_2 , and A_3 is:

$$\mathbf{A_1 \cap A_2 \cap A_3 = \{2\}}$$

That is, we pick the elements common to *every set*.

Note we emphasized the phrases "*every* set" and "in *at least one* of the sets". This *every* vs. *at least one* dichotomy is *most* important in our final proof.

The next step is to - quote - "simplify" - unquote - our notation.

"Life is really simple, but we insist on making it complicated."
- Attributed to Confucius

OK. Now let's define a *class* of sets, S, in terms of our earlier sets:

$$S = \{ A_1 , A_2 , A_3 \}$$

And to make it even simpler, we represent the union as:

$$\cup S = \cup\ A_i \text{ where } i = \{ 1 , 2 , 3 \}$$

We can then write the *union* with admirable brevity:

$$\cup S = \{ 1 , 2 , 3 , 4 , 5 , 6 , 8 , 14 , 20 \}$$

And for the intersection we write:

$$\cap S = \cap\ A_i \text{ where } i = \{ 1 , 2 , 3 \}$$

... which for our class of sets is:

$$\cap S = \{ 2 \}$$

OK. Now let's create ***general definitions*** for union and intersection. In English this is:

> **The union of a class of sets is the set of elements that is in at least one of the sets.**

... and

> **The intersection of a class of sets is the set of elements that is in every set.**

Notice these generalized definitions are almost the same. The only difference is that *union* talks about *at least one* set in the class. The *intersection* talks about *every* set.

Again we have the *at least one* / *every* dichotomy.

"Writing in English is like throwing mud at a wall."

- Somebody But Probably Not Joseph Conrad

The last two definitions for union and intersection may seem precise but actually are not. They are "informal" as mathematicians would say. To get further into our proof, we need to replace our English with a completely ***formal*** language. That doesn't mean you speak politely. It means you use ***symbols*** instead of words and ***formulas*** instead of sentences.

So obviously we need to to start defining symbols. And so we begin with:

∪	**=**	**Union**
x ǀ	**=**	**The set of all x such that**
∃	**=**	**For at least one (For some)**
∈	**=**	**is an element of**

And now we can write:

$$\cup S = \{ x | \exists A_i \in S, x \in A_i \}$$

So by checking the table, we read this as:

> **The union of a class of sets, $\cup S$, is the set of all x such that for at least one set A_i that is an element of S, x is an element of A_i.**

In other words, the element x only has to be in ***at least one*** of the sets. Then it is a member of the union of the sets.

Again note we used the phrase ***at least one***.

And now we do the same thing for intersection. That is, we take the definitions:

$\cap$	=	**Intersection**
x |	=	**The set of all x such that**
$\forall$	=	**For every (For all)**
$\in$	=	**is an element of**

... and we get:

$$\cap S = \{ x | \forall A_i \in S, x \in A_i \}$$

... which you can see is verbalized as:

The intersection of a class of sets, $\cap S$, is the set of all x such that for every set A_i that is an element of S, x is an element of A_i.

In other words, the element x must be in ***every*** one of our sets. Then x is a member of the intersection of the sets.

Once more these more formalized definitions differ only in the phrases "***for at least one***" and "***for every***". And these ***are*** the definitions you find if you go to the more advanced texts.

But there's one thing the textbooks don't mention. Even these "rigorous" definitions leave out quite a bit of the true logical language. When we include the logic, we will see the two definitions becomes quite a bit different.

"It is not logical."
- *The Wrath of Khan*, 1982

OK. At this point it may look like we're heading off to where no man has gone before or at least getting sidetracked. But actually we're going to start talking about how we logically express the concepts "for at least one" (or "some") and "for every" (or "all"). Specifically, we'll ask just what does it mean when we say:

All P's are Q's.

... and

Some P's are Q's.

First we'll look at:

All P's are Q's.

Logicians tell us "All P's are Q's" is defined by:

All P's are Q's	**=**	**For every thing in the world, if it has property P then it has property Q**

Now we can write this entirely without words - that is symbolically - if we define the symbols:

$\forall$	**=**	**"For all"** **"For every"**
$\rightarrow$	**=**	**"If-Then"**
P(x)	**=**	**x has property P**
Q(x)	**=**	**x has property Q**

... and write:

$$\forall x, P(x) \rightarrow Q(x)$$

If we switch back to the stilted English - sometimes called "Loglish" - we're simply saying

For every x, if x is P, then x is Q.

In other words, every x that has a property P will also will have property Q. And you can see that this is indeed saying "All P's are Q's".

But ***is*** this the best way to say, "All P's are Q's"? For instance, could we define a new symbol:

$$\wedge = \text{AND}$$

... and write:

$$\forall x, P(x) \wedge Q(x)$$

That is, we say:

For every x, x is P AND x is Q.

Unfortunately, this doesn't work. This sentence actually means,

"Everything in the world is both P AND Q."

Certainly not what we want to say.

So if we want to say "All P is Q", we have to stick to the "If-Then" structure.

OK. But ***now*** suppose we want to want to say:

Some P's are Q's.

Now it's tempting - and common - to believe we just take the "All P is Q" structure and write:

$$\exists x, P(x) \rightarrow Q(x)$$

... where

"For at least one"
$\exists$ = or
"For some"

But as we'll show a bit further down, you can't do this. Instead we ***do*** have to use an "**AND**" type structure. That is we have to write:

$$\exists x, P(x) \land Q(x)$$

... which means:

For at least one x, x is P and x is Q.

Note that this does ***not*** not mean everything in the world is P and Q. Instead, it simply means ***some things*** are both P and Q - which is the same as "Some P's are Q's."

But why ***can't*** we use the "If-Then" structure? It seems like that should work.

Well, it doesn't. And to understand why, we have to turn to the details of what makes an "**AND**" statement TRUE and what makes an "If-Then" statement TRUE. You might call this:

The Great "If-Then" vs. "AND" Debate

OK. Suppose it's raining. Then you might tell one of your friends:

Don't go outside or you will get wet.

Now notice how this means the same thing as:

If you do go outside, you will get wet.

So if we define two new symbols as:

$$\neg \;=\; \textbf{Not}$$
$$\vee \;=\; \textbf{Or}$$

... we write the equations:

$$P \rightarrow Q \quad = \quad \neg P \vee Q$$

("If P, Then Q" = "Not P Or Q")

Now in logic, a sentence containing the word "**OR**" is TRUE if at least one of the parts is TRUE. For instance, if I say:

George Washington was the first President

... and

Thomas Jefferson was the 44th President.

... then the statement:

George Washington was the first President OR Thomas Jefferson was the 44th President.

... is TRUE even though Tom was the third president.

So a sentence like

$$\mathbf{P \lor Q}$$

... is TRUE unless ***both*** P ***and*** Q are FALSE.

Logicians usually put the TRUE or FALSE values of sentences in ***Truth Tables***. The Truth Table for "**OR**" sentences, $\lor$, is:

Truth Table: OR ($\lor$) Statements

P	Q	P $\lor$ Q
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

Now what about ***this*** sentence:

$$\mathbf{\neg P \lor Q}$$

Well, if P is TRUE, then ¬P (that is, "Not P") must be FALSE and vice versa. And so from our definition of $\lor$, we get the following Truth Table for $\mathbf{\neg P \lor Q}$.

Truth Table: ¬P ∨ Q

P	¬P	Q	¬P ∨ Q
TRUE	FALSE	TRUE	TRUE
TRUE	FALSE	FALSE	FALSE
FALSE	TRUE	TRUE	TRUE
FALSE	TRUE	FALSE	TRUE

In other words, the statement "**¬P ∨ Q**" is FALSE ***only if*** P is TRUE and Q is FALSE

But remember. **"¬P ∨ Q"** is the same as the "If-Then" statement, **"P → Q"**. The full Truth Table of the "If-Then" statement is therefore:

Truth Table: If-Then Statements

P	Q	P → Q
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	TRUE
FALSE	FALSE	TRUE

So an "If-Then" statement is FALSE only if the "If" part - called the ***antecedent*** - is TRUE and the "Then" part - called the ***consequent*** - is FALSE.

Which brings us to a crucial point in our proof. That is:

An "If-Then" Statement is always TRUE if the "If" part is FALSE.

The idea that an "If-Then" statement must be TRUE if the "If" part is FALSE causes some students consternation. But it actually works out OK. If you wish, you can see a more detailed derivation of the "If-Then" Truth Table and how it's the same as the "Not-Or" Table at http://www.coopertoons.com/education/emptyclass_intersection/education/ifthenlogic/ifthen1.html. But if that explanation still leaves you wondering, as an American President once said, "Trust me."

Now we'll shift gears once more. Let's take a look at the ***correct*** expression for saying "Some P is Q". That is:

Some P is Q = **$\exists$x, P(x) $\wedge$ Q(x)**
(For at least one x, P AND Q)

Remember, an "**AND**" statement, $\wedge$, is TRUE ***only*** when ***both*** P and Q are TRUE. The full "**AND**" Truth Table is:

Truth Table: AND ($\wedge$) Statements

P	**Q**	**P $\wedge$ Q**
TRUE	**TRUE**	**TRUE**
TRUE	**FALSE**	**FALSE**
FALSE	**TRUE**	**FALSE**
FALSE	**FALSE**	**FALSE**

So for "**$\exists$x, P(x) $\wedge$ Q(x)**" to be TRUE we must find that there is ***at least one x*** which has both property P and property Q. We can then legitimately claim that "Some P is Q" is TRUE.

But what if we look at the ***incorrect*** expression for "***Some*** P is Q"? That is, if we write:

$$\mathbf{\exists x, P(x) \rightarrow Q(x)}$$

Why can't we use that instead?

Well, let's look to see what makes the statement TRUE.

From the "If-Then" Truth Table, we see that "$\mathbf{\exists x, P(x) \rightarrow Q(x)}$" is only FALSE if P(x) is TRUE and Q(x) is FALSE. So there are a lot of ways to make this statement TRUE.

For instance, if a particular x has both property P and property Q, then the statement "$\mathbf{P \wedge Q}$" is TRUE.

But suppose x doesn't have property P at all. Then P(x) is FALSE. So the entire statement "$\mathbf{\exists x, P(x) \rightarrow Q(x)}$" is always TRUE. Then it doesn't matter if Q(x) is TRUE or FALSE.

We see, then, that a statement like "$\mathbf{\exists x| P(x) \rightarrow Q(x)}$" gives us too many options for **x**. In fact, such statements are pretty much worthless in logic. It certainly doesn't mean "Some P are Q".

So in the end we must conclude that the ***correct*** sentences are:

$$\textbf{Some P is Q} \quad = \quad \mathbf{\exists x, P(x) \wedge Q(x)}$$

... and ...

$$\textbf{All P is Q} \quad = \quad \forall x, P(x) \rightarrow Q(x)$$

When we want to talk about ***all things*** we ***must*** use an "If-Then" statement. And if we talk about ***at least one thing***, we ***must*** use the "**AND**" statement.

So ***now*** we can begin.

A Good Start

We are now going to define the concepts of the intersection and union of sets in a most rigorous manner. But by now you shouldn't have much trouble with our logical lingo.

For sets A_i contained in a class ***S***, we define the ***union*** of the class, $\cup S$ as:

$$\cup S = \{\, x \mid \exists A_i, (A_i \in S) \wedge (x \in A_i) \,\}$$

This is stated as:

The union of the sets in class *S*, ∪*S*, is the set of all

elements, x, such that for at least one set A_i, A_i an element of *S* AND x is an element of A_i

As we have now repeated ***ad nauseam***, we have to use the "**AND**" form since we are talking about ***at least one*** of the subsets, A_i.

But also notice how this works fine to define the union of the sets. We pick out a set of elements (x) where it is true that A_i is an element of ***S AND*** x is in ***at least one*** A_i.

In other words, all elements in the sets in ***S*** makes up the union of the sets.

So far so good.

Now we can define ***intersection*** as:

$$\cap \mathbf{S} = \{ \mathbf{x} | \forall \mathbf{A_i}, (\mathbf{A_i} \in S) \rightarrow (\mathbf{x} \in \mathbf{A_i}) \}$$

This is stated as:

The intersection of the sets in class *S*, $\cap S$, is the set of all elements, x, such that for every set A_i, IF A_i is an element of *S* THEN x is an element of A_i

And yes, because we use "for every", ∀, we have to use the "If-Then" form.

Again notice what we've written works fine for our notion of intersection. We select the x's where then x is in ***every*** set as long as A_i is in ***S***.

In other words, the intersection is the x's common to ***every*** set.

"I love to talk about nothing ... It's the only thing I know anything about."

- Oscar Wilde (*An Ideal Husband*)

And ***finally*** we can arrive at the answer.

Remember our set ***S*** is a ***class*** of sets.

$$\boldsymbol{S = \{A_1 , A_2 , A_3\}}$$

But now let's take all the sets out.

$$\boldsymbol{S = \{ \}}$$

This leaves us with the ***empty class***, which we represent with the symbol:

$$\boldsymbol{\Phi}$$

That is, $\boldsymbol{\Phi}$ is a class of sets which just happens to contain no sets.

So how do we define the ***union*** of $\boldsymbol{\Phi}$, $\boldsymbol{\cup\Phi}$?

Actually it's easy. We simply substitute $\boldsymbol{\Phi}$ for ***S*** in our definition.

That is, since we defined:

$$\cup S = \{ x \mid \exists A_i, (A_i \in S) \wedge (x \in A_i) \}$$

.. the union of **Φ**, $\cup\Phi$, is just:

$$\cup\Phi = \{ x \mid \exists A_i, (A_i \in \Phi) \wedge (x \in A_i) \}$$

But there's one important thing to remember.

Φ is the ***empty class***. It contains ***no sets***.

In other words, the statement:

$$A_i \in \Phi$$

... is always FALSE.

And also remember. If you have an "**AND**" statement, it is FALSE as long as ***one*** of the parts is FALSE.

So our somewhat long ***defining sentence*** for $\cup\Phi$:

$$\cup\Phi = \{x \mid \exists A_i,\ (A_i \in \Phi) \ \wedge \ (x \in A_i)\}$$

... now becomes ...

$$\cup\Phi = \{x \mid \ \text{FALSE} \ \wedge \ \text{(Doesn't\} Matter)}$$

... which collapses to the rather strange - but correct - definition:

$$\cup\Phi = \{ x \mid \text{FALSE} \}$$

That is, there are no x's which make the ***defining statement*** TRUE. So the union of the empty ***class*** is indeed the empty ***set***.

$$\cup\Phi = \varnothing$$

And for ***intersection***?

Again we take our general definition, for intersection:

$$\cap S = \{ x \mid \forall A_i, (A_i \in S) \rightarrow (x \in A_i) \}$$

... and substitute **Φ** for ***S***:

$$\cap\Phi = \{ x \mid \forall A_i, (A_i \in \Phi) \rightarrow (x \in A_i) \}$$

And ***here*** at long last we reach the answer.

We mentioned that an "If-Then" statement is FALSE only when the "If" part is TRUE, and the "Then" part is FALSE.

But that means that if the "If" part is always FALSE, the whole statement is always TRUE!

And note that in the statement:

$$\mathbf{A_i \in \Phi \rightarrow x \in A_i}$$

... the "If" part:

$$\mathbf{A_i \in \Phi}$$

... is indeed always FALSE. Again there are no sets in the empty class.

Therefore our defining sentence for the ***intersection of the empty class***:

$$\mathbf{\cap\Phi = \{x \mid \forall A_i,\ (A_i \in \Phi) \rightarrow (x \in A_i)\}}$$

... now becomes ...

$$\mathbf{\cap\Phi = \{x \mid \text{FALSE} \rightarrow \text{(Doesn't\} Matter)}}$$

But this statement always TRUE! And the intersection of **Φ** ends up being defined by

$$\cap\mathbf{\Phi} = \{\ x \mid \text{TRUE}\ \}$$

So any x, that is, anything in the world - even in ***the whole universe*** - makes the statement TRUE. ***Anything*** is in the intersection of Φ. That is, the intersection of the ***empty class*** is the ***universal set***.

$$\cap\mathbf{\Phi} = \mathbf{U}$$

"Nikita! What's going on?"

-Laventry Beria to Nikita Khrushchev, June 26, 1953

It's one thing to - somewhat mechanically - show that by certain definitions you can produce amusing paradoxes. But the real ***student*** wants to ***understand*** what's going on.

In particular, if we're talking about the intersection of the empty class being everything, where did all the elements come from?

We must re-emphasize that we are referring to the ***empty class***. That is, we are talking about a set whose elements are *other sets*.

So let's look again at our general definition of ***intersection***:

$$\cap S = \{\ x \mid \forall A_i, (A_i \in S) \rightarrow (x \in A_i)\ \}$$

And of course, there is ***nothing that requires the individual sets, A_i in our class, S, to be empty***.

And now for the ***really*** strange part.

Even when we require the ***class*** to contain ***no sets*** - that is, we create **Φ** - we still define its intersection as:

$$\cap \boldsymbol{\Phi} = \{ \mathbf{x} \mid \forall \mathbf{A_i}, (\mathbf{A_i} \in \boldsymbol{\Phi}) \rightarrow (\mathbf{x} \in \mathbf{A_i}) \}$$

Note that the individual sets, A_i, can themselves ***still have elements and are still part of our definition!*** The non-empty sets, A_i, and their individual elements, x, are still there. So if the defining statement is TRUE, we have the x's - even if the class, **Φ**, is empty!

At this point, we can raise a point that is so subtle that it usually passes without comment. But we'll raise the point anyway.

We said that the entire statement:

$$(\mathbf{A_i} \in S) \rightarrow (\mathbf{x} \in \mathbf{A_i})$$

..is TRUE if ...

$$A_i \in S$$

... is FALSE. That is, the "If" part can be FALSE and so the "Then" part, $\mathbf{x} \in \mathbf{A_i}$, can be TRUE or FALSE as we please.

So we wonder.

Can't we simply pick a non-empty set A_i that is ***not*** in $\boldsymbol{S}$? Won't that make

$$\mathbf{A_i \in \boldsymbol{S} = FALSE}$$

.. and we can then pick any x in the world? So in that case even if $\boldsymbol{S}$ is ***not*** empty, wouldn't the intersection of $\boldsymbol{S}$ be the Universal Set?

Actually no. Remember ***we*** are the ones to pick out which sets, A_i, we want to determine if they intersect. It's our arbitrary decision. Therefore these sets ***define*** the class we call $\boldsymbol{S}$. As long as we are talking about *S as a non-empty class, then the expression:*

$$\forall \mathbf{A}(A_i \in \boldsymbol{S})$$

... is - and pardon us if we shout - ***TRUE BY DEFINITION!***

What we are doing, then, is limiting our ***Universe of Discourse*** as the class $\boldsymbol{S}$. So there are no sets in our universe not in $\boldsymbol{S}$. Therefore the defining equation of intersection:

$$\{x \mid \forall A_i, (A_i \in S) \rightarrow (x \in A_i)\}$$

... with the IF-part being TRUE by definition, the whole IF-THEN statement can only be TRUE if the THEN-part is also TRUE. Ergo, the set of **x** that make up the intersection is only defined by the **x** values that are in every A_i that we specify.

Therefore the defining equation of intersection:

$$\{ \mathbf{x} | \forall \mathbf{A_i}, (\mathbf{A_i} \in \boldsymbol{S}) \rightarrow (\mathbf{x} \in \mathbf{A_i}) \}$$

Ergo, the set of **x** that make up the intersection is only defined by the **x** values that are in every A_i that we specify.

It's only when our class ***S*** is the empty class, **Φ** that the "If" part of the statement is ***always*** FALSE. This makes the whole "If-Then" statement always TRUE for any x. So we have removed any restrictions on what sets we have to choose from. We can now pick every element in every set in the world - and we end up with the universal set.

"'Twas strange, 'twas passing strange."

- William Shakespeare (*Othello*)

All of what we've just written seems strange. But ***strange*** does not mean ***incorrect***.

We've left one thread hanging - which will also seem strange. What about the books that say the ***intersection*** of the empty ***set*** is the ***empty*** set.

After all you'll see things like

For all sets, the union of the empty set is the set itself, and the intersection of the empty set is the empty set.

... which symbolically is:

$$\forall A,\ A \cup \varnothing = A$$
$$\forall A,\ A \cap \varnothing = \varnothing$$

Here we're saying ***for all*** and using $\forall$. That means ***all*** sets - ***not all except one***.

The empty set is indeed a set. So from our equations it should also be true that:

$$\varnothing \cup \varnothing = \varnothing$$
$$\varnothing \cap \varnothing = \varnothing$$

But hold on there, pilgrim. We've just got finished showing the *intersection* of the empty set is the *universal set*.

No, we didn't. We showed the ***intersection*** of the empty ***class*** is the universal set.

Now although you will read that there is only ***one*** empty set - the empty set is unique - we can show that there really is no conflict when talking about an ***empty class*** vs. an ***empty set***. And the distinction will clear up the confusion.

Consider an oddball set called ***T*** which we define as:

$$T = \{ \varnothing \}$$

Now notice. ***T*** is a set that contains a set. Although ∅ contains no elements, ***T*** does. It's sole element is the empty set. So ***T*** is not the empty class.

Now what does it mean if we talk about the union or intersection of ***T***? Well the union and intersection of a class is the union and intersection of the sets which are elements of the class. So when talking about the union and intersection of ***T***, we are talking about the union and intersection of the ***empty set.***

To determine what the new union and intersection will be, we first return to our definition for ***union***.

$$\cup S = \{ x| \ \exists A_i, (A_i \in S) \wedge (x \in A_i) \}$$

But we now insert our new class, ***T***, into the definition. We now have:

$$\cup T = \{ x| \ \exists A_i, (A_i \in T) \wedge (x \in A_i \}$$

So we ask. Is the first part:

$$A_i \in T$$

... TRUE or FALSE?

Well, is TRUE. After all, we defined ***T*** so that its contains the empty set, ∅.

$$T = \{ \varnothing \}$$

So there is a set, A_i in $\boldsymbol{T}$. Although it's the empty set:

$$\mathbf{A_i = \varnothing}$$

... and so ...

$$\mathbf{A_i \in \boldsymbol{T}}$$

... is indeed TRUE. $\varnothing$ may contain no elements, but $\boldsymbol{T}$ does.

On the other hand, we also know that the second part of the "If-Then" statement:

$$\mathbf{x \in A_i}$$

... is the same thing as:

$$\mathbf{x \in \varnothing}$$

... which is FALSE. After all, $\varnothing$ has no elements.

So we can specifically rephrase our defining sentence as:

$$\cup \boldsymbol{T} = \{ x| \underbrace{(\varnothing \in \boldsymbol{T})}_{\text{TRUE}} \underbrace{\wedge}_{\text{AND}} \underbrace{(x \in \varnothing}_{\text{FALSE}} \}$$

Since this is an "**AND**" statement and one of the parts is FALSE, the whole statement is FALSE.

$$\mathbf{\cup \boldsymbol{T} = \{x| \text{ FALSE } \}}$$

... and so ...

$$\cup T = \varnothing$$

So the union of the empty ***set*** with itself is indeed the empty set. Like what we saw before.

But now things will get a bit different when we look at the ***intersection*** of ***T***, $\cap T$.

Remember the general definition for intersection is:

$$\cap \mathbf{S} = \{\, \mathbf{x} \mid \forall \mathbf{A_i}, (\mathbf{A_i} \in S) \rightarrow (\mathbf{x} \in \mathbf{A_i}) \,\}$$

... and we plug in the formulas for ***T*** and $\varnothing$:

$$\cap \mathbf{S} = \{\, \mathbf{x} \mid \forall \mathbf{A_i}, (\mathbf{A_i} \in T) \rightarrow (\mathbf{x} \in \mathbf{A_i}) \,\}$$

... which when we substitute $\varnothing$ for A_i we get:

$$\cap \mathbf{S} = \{\, \mathbf{x} \mid (\varnothing \in T) \rightarrow (\mathbf{x} \in \varnothing) \,\}$$

Now look at the logic. As before we have:

$$(\varnothing \in T) \quad = \quad \text{TRUE}$$

... and ...

$$(x \in \varnothing) \;=\; \text{FALSE}$$

So going back to our ***defining sentence*** we have:

$$\boldsymbol{\cap T = \{\, x| \quad (\varnothing \in T) \quad \rightarrow \quad (x \in \varnothing\,\}}$$

IF TRUE THEN FALSE

From the Truth Table of the "If-Then" sentence we remember:

$$\mathbf{(TRUE \rightarrow FALSE) = FALSE}$$

...and see that our ***defining sentence*** for intersection of the empty ***set*** - unlike that for the empty ***class*** - is ***always*** FALSE. Again the more complex sentence simplifies, this time to:

$$\boldsymbol{\cap T = \{x \mid \text{FALSE}\,\}}$$

The intersection of the empty set with itself is indeed ∅.

To summarize our conclusions, for a class ***T*** defined as:

$$\boldsymbol{T} = \{ \varnothing \}$$

... we find that ...

$$\bigcup T = \varnothing$$
$$\bigcap T = \varnothing$$

... even though earlier we found:

$$\bigcup \Phi = \varnothing$$
$$\bigcap \Phi = \mathrm{U}$$

Are you saying, then, that there are really ***two*** empty sets? One that does that not contain sets and the other does not contain objects?

No, the empty set is indeed unique. What produced our apparently contradictory findings is that we are dealing with two distinct definitions. In one case - where the intersection of the empty set (or class) is the universal set - we are talking about the intersection of sets in a class devoid of sets.

In the other case, though, when we get the intersection of the empty set to be the empty set, we are talking about the intersection of a non-empty class whose sole element is the empty set, $\varnothing$. The two concepts are ***not*** the same nor contradictory. We are simply talking about different types of intersection. We shouldn't be surprised if the two definitions give us different answers.

Which they do.

Blowing It Off

We can now understand why the topic of the union and intersection of the empty class or set is pretty much blown off by authors of math textbooks. And why they just say you should "convince yourself" or "define" the statements to be true.

To explain the union and intersection of the empty class as a rigorous proof, the authors would have to wander into the field of symbolic logic. This would then force the student to grapple with two specific points of logic - and points that can be particularly confusing.

- The definitions of ***union*** and ***intersection*** require an understanding that the logical sentences, "***Some P's is Q***" and "***All P's are Q***" are actually completely different logical expressions. The former sentence - defining ***union*** - requires an **AND** construction and the latter - defining ***intersection*** - needs an **If-Then** structure.
- The second point of confusion arises from **If-Then** sentences always being TRUE if the **If** part - the ***antecedent*** - is FALSE. Teachers have been struggling to explain this characteristic to their students literally for millennia. But it is essential to understanding why the intersection of the empty set is the universal set.

A modest CooperToons suggestion is that the authors of the textbooks on set theory should simply cut out the "convince yourself" nonsense as if what they're saying is something any dunderhead should see immediately. Instead they should just admit the union and intersection of the empty class seem strange but are due to subtle points of logic. Maybe they can convince themselves this is a better way.

But in any case, we have ***finally*** seen we ***can*** write with perfect correctness:

$$\bigcup \boldsymbol{\Phi} = \varnothing$$

... and

$$\bigcap \boldsymbol{\Phi} = \mathbf{U}$$

And ***now*** we can say:

"I understand."

References

Introduction to Logic, Patrick Suppes, Van Nostrand, 1957. A classic and popular text that teaches symbolic logic. No truth trees but a lot of information nowadays often omitted in today's textbooks. Specifically this book details the differences between "Some P are Q" and "All P or Q" and discusses why the first must use "**AND**" and the other "**If-Then**".

Set Theory and Logic, Robert Stoll, Freeman, 1963. A good enough text but there is a bit too much of the "reader should convince themselves" stuff - including "It is left to the reader to convince himself that the defining property at hand is satisfied by any object whatever" when talking about the intersection of the "empty collection." Given the fact that Professor Suppes in his book specifically warns the student that confusing the ***All/Every*** sentences is easy, and Philo of Megara was having trouble arguing for the modern "**If-Then**" sentences thousands of years before, the "convince yourself" school of pedagogy can be quite irritating for those trying to combat ignorance and superstition.

Formal Logic: Its Scope and Limits, Richard Jeffrey, McGraw-Hill (1981). Richard was a great popularizer of formal logic and an exponent of using truth trees to find validity or non-validity in arguments. With truth trees - which are stick type diagrams - you either show the argument valid or automatically arrive at counterexamples. Truth trees are really slick.

Unfortunately, Richard doesn't go into the detail of the classical logic as much as Patrick does in his book. So both books of these gentlemen are a good additions to the library of the armchair logician.

Julius Caesar, William Shakespeare, Folger Shakespeare Library / Simon and Schuster, 1995.

Othello, William Shakespeare, Folger Shakespeare Library / Simon and Schuster, 1993.

Manhunt of Mystery Island, Spencer Bennet (Director), Ronald Davidson (Producer), Republic Studios (1945). A nice respite from reading mathematical or logical textbooks. Or surfing the internet for that matter.

Selected Poetry of Ogden Nash, Ogden Nash, Black Dog and Leventhal, 1995.

Khrushchev: The Man and His Era, William Taubman, W. W. Norton and Company, 2003.

Websites on the Fount of All Knowledge. You can find some chatrooms on the topic of this essay, but as often happens the sites can vanish with sad frequency. In any case, an honest CooperToons opinion is they do not really explain the situation in a way to satisfy the layman or student, not even the informed layman or student. True, some will provide the definitions of intersection in the "If-Then" (or the equivalent "Whenever") formula and the union in the "**AND**" format, but generally in an abbreviated and informal manner. Also the authors seem to think the reader readily understands why you use the different formulas. But as Patrick pointed out in *Introduction to Logic*, confusing the "All" and "For at least one" formulas is quite common for the fledgling student of logic.